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ON THE LINEAR PROBLEM OF A SUPERSONIC FLOW OF A VISCOUS GAS PAST A VIBRATOR*

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Authors of /1,2/ use the framework of the linear theory to investigate a supersonic boundary layer with selfinduced pressure near a triangular wall oscillating harmonically with a small amplitude. Solution of the equations for the inner sublayer gave rise to a dispersion relation the roots of which determined the perturbations travelling up and down the stream.

The present paper deals with the problem of supersonic gas flow past a vibrator the oscillation amplitude of which increases with time, in the linear approximation. A dispersion relation arising in this case is studied. It is shown that the number of eigen solutions of the problem depends essentially on the character of the vibrator oscillations.

1. Formulation of the problem. Let us consider a flow past a flat, thermally insulated plate of length L, transforming into a triangular oscillating projection, i.e. a vibrator, and ending with a flat plate of length $O(L)$. Let t be time, x and y the Cartesian coordinates, u and v the velocity vector components, ρ density, p the pressure and λ_1 the first viscosity coefficient. Assuming the velocity of the impinging stream p be equal to U_∞ $(M_\infty > 1)$, we introduce the small parameter $\varepsilon = \mathrm{Re}_1^{-1/8}$ $(\mathrm{Re}_1 = \rho_\infty U_\infty L/\lambda_1)$. We shall assume that the Prandtl number is unity and $\lambda_1/\lambda_{1\infty} = CT/T_\infty$.

Let us select a longitudinal dimension of the oscillating segment $O(\varepsilon^3 L)$, the oscillation frequency $O(\varepsilon^{-2}U_\infty/L)$ and the amplitude $O(\varepsilon^5 L)$. We describe the flow using the boundary layer equations with selfinduced pressure /1−3/. The oscillating part of the wall is described, in terms of dimensionless coordinates of the inner sublayer, by the equation

$$y_w = \sigma f(x)\exp(\omega_1 t)\cos(\omega_2 t) \tag{1.1}$$

Here the parameter $\sigma \ll 1$, $\omega_1 > 0$ and $\omega_2 > 0$ are the dimensionless frequencies and the motion is studied over the interval contained between some infinitely distant instant of time and $t = 0$. The function $f(x)$ defines the triangular form with parameters a, b and h

$$f(x) = \begin{cases} 0, & x \leqslant 0 \\ hx/b, & 0 \leqslant x \leqslant b \\ h(a-x)/(a-b), & b \leqslant x \leqslant a \\ 0, & x \geqslant a \end{cases} \tag{1.2}$$

The problem is linearized with respect to the Blasius equation by substituting into the starting equations for the boundary layer with selfinduced pressure, the expansions of the unknown functions into series in terms of the small parameter σ

$$u = y + \sigma u_1 + \ldots,\quad v = \sigma v_1 + \ldots,\quad p = \sigma p_1 + \ldots$$

The following problem arises here for the first approximation functions:

$$\frac{\partial u_1}{\partial t} + y\frac{\partial u_1}{\partial x} + v_1 = -\frac{\partial p_1}{\partial x} + \frac{\partial^2 u_1}{\partial y^2},\quad \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \tag{1.3}$$

$$\frac{\partial p_1}{\partial y} = 0;\quad \rho = 1;\quad u_1 \to 0,\quad p_1 \to 0\quad (x \to -\infty)$$

$$u_1 \to -\int_{-\infty}^{x} p_1(t,x)\,dx\ (y \to \infty);\quad u_1 = -y_{w1},\quad v_1 = \frac{\partial y_{w1}}{\partial t}\quad (y = 0)$$

$$y_{w1} = f(x)\exp(\omega_1 t)\cos(\omega_2 t)$$

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We note that to obtain the conditions of adhesion given above, we must consider an auxilliary subregion near the wall the characteristic dimension of which is $y_1 = y/\sigma$, since the condition of linearization $u_1 = O(1)$ is violated as $y \to 0$.

To simplify the arguments, we introduce the complex frequency $W = \omega_1 + i\omega_2$ and consider the functions of velocity and pressure as complex functions of the real variable, remembering that only their real parts have any physical meaning. Let $u_1 = U(x, y)\exp(Wt)$, $p_1 = P(x)\exp(Wt)$. Eliminating the function v_1 from (1.3), we obtain

$$\frac{\partial^3 U}{\partial y^3} - y\frac{\partial^2 U}{\partial x \partial y} - W\frac{\partial U}{\partial y} = 0;\quad U \to 0,\ P \to 0 \quad (x \to -\infty) \tag{1.4}$$

$$U \to -\int_{-\infty}^{x} P(x_1)\,dx_1 \ (y \to \infty);\quad U = -f(x) \quad (y = 0)$$

We construct the solution of (1.4) in the usual manner, using the Fourier transformation

$$U(k, y) = \int_{-\infty}^{\infty} e^{-ikx} U(x, y)\,dx$$

The transformed equation (1.4) represents the well known Airy equation the solution of which, expressed in terms of the Airy function Ai and satisfying all boundary and limiting conditions of (1.4), has the form

$$P(k) = ikf(k)\frac{d\mathrm{Ai}(\Omega)}{d\Omega}\left[\frac{d\mathrm{Ai}(\Omega)}{d\Omega} + (ik)^{1/3}\int_{\Omega}^{(ik)^{1/3}\infty} \mathrm{Ai}(z)\,dz\right]^{-1} \tag{1.5}$$

$$\Omega = \frac{W}{(ik)^{2/3}},\quad f(k) = \frac{h}{b(ik)^2}\left[1 - \frac{a}{a-b}e^{-ikb} + \frac{b}{a-b}e^{-ika}\right]$$

To separate the single-valued branch of the function $P(k)$, we make a cut in the complex plane $k = k_1 + ik_2$ along the positive imaginary semiaxis /2–5/. If $-3\pi/2 \leqslant \arg k \leqslant \pi/2$, then the integral in (1.5) can be written in the form

$$\int_{\Omega}^{(ik)^{1/3}\infty} \mathrm{Ai}(z)\,dz = \int_{0}^{\infty} \mathrm{Ai}(x)\,dx - \int_{0}^{\Omega} \mathrm{Ai}(z)\,dz = I_0 - I_1(\Omega)$$

The poles of the analytic function $P(k)$ are determined by the following dispersion equation:

$$\Phi(W, \Omega) \equiv \Omega^2\frac{d\mathrm{Ai}(\Omega)}{d\Omega} + W^2(I_0 - I_1(\Omega)) = 0 \tag{1.6}$$

$$\arg W - {}^2/_3\pi < \arg\Omega < \arg W + {}^2/_3\pi\ (0 \leqslant \arg W \leqslant \pi/2)$$

2. Zeros of the dispersion equation. Consider the asymptotic behavior of the roots of the equation (1.6) as $|\Omega| \to \infty$ (the value of $|W|$ is finite). From (1.6) it follows that the negative real semiaxis ($\arg\Omega = \pi$) appears within the domain of admissible values of Ω, if $\pi/3 \leqslant \arg W \leqslant \pi/2$, and emerges outside its boundaries when $0 \leqslant \arg W < \pi/3$. Let $|\Omega| \gg 1, \pi/3 \leqslant \arg W \leqslant \pi/2$. We shall use the known expressions for the derivative and integral of the Airy function in the region containing the semiaxis $\arg\Omega = \pi$. Substituting these expressions into (1.6), we obtain in the first approximation

$$\Phi(W, -\Omega) \approx -\pi^{-1/2}\Omega^{5/4}\cos\left(\frac{2}{3}\Omega^{3/2} + \frac{\pi}{4}\right) + W^2\int_{-\infty}^{\infty} \mathrm{Ai}(x)\,dx = 0 \tag{2.1}$$

$$|\arg\Omega| < {}^2/_3\pi$$

Equations (1.6) were solved on a digital computer for the range of values $|\Omega| = O(1)$, by expanding the derivative and integral of the Airy function into series. The contour lines $\mathrm{Re}[\Phi(W, \Omega)] = 0$ and $\mathrm{Im}[\Phi(W, \Omega)] = 0$ were constructed in the complex plane for fixed values of W and the points of intersection of these contours gave the required solutions /2/. Figures 1 and 2 depict the patterns obtained for $|W| = 3$, for $\arg W = \pi/2$ and $\arg W = \pi/6$, respectively.

The solid and dashed lines depict $\mathrm{Re}[\Phi(W, \Omega)] = 0$ and $\mathrm{Im}[\Phi(W, \Omega)] = 0$, and the shading indicates the part of the plane for which the relation (1.6) does not hold.

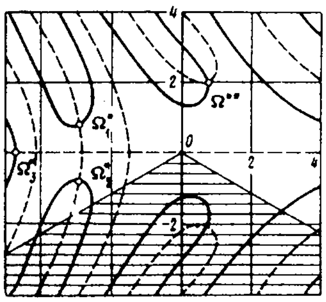

Fig.1

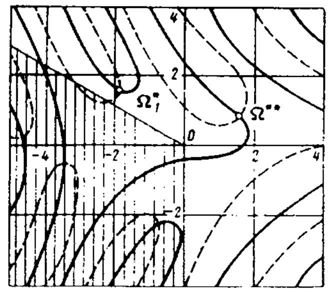

Fig.2

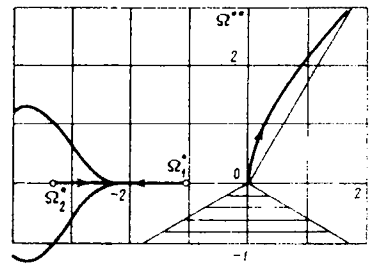

Fig.3

We have found that, irrespective of the value of $\arg W$, we can always find a unique root Ω^{**} of the dispersion equation in the first quadrant of the plane Ω. The pattern of distribution of the zeros Ω_n^* $(n = 1, 2 \ldots)$ in the region $\mathrm{Re}(\Omega) < 0$ depends on the value of $\arg W$. If $\pi/3 \leqslant \arg W \leqslant \pi/2$, then the distribution tends, with increasing distance from the coordinates origin, to the asymptotic distribution described above. The solutions have the corresponding roots k^{**} $(\mathrm{Im}\, k^{**} < 0)$ and k_n^* $(\mathrm{Im}\, k_n^* > 0)$ in the plane k under consideration. In the case when $\pi/3 \leqslant \arg W \leqslant \pi/2\, k_n^*$, they represent an infinite sequence with the condensation point at the zero, the sequence defined for $n \gg 1$ by the relations (2.2). If $0 \leqslant \arg W < \pi/3$, then the number of solutions k_n^* is finite although it may change depending on the value of $|W|$.

Figure 3 illustrates the displacement of the roots in the Ω plane relative to values of $|W|$ when $\arg W = \pi/2$. The arrows indicate the direction of increasing $|W|$ from its zero value. We see that for small values of $|W|$ two roots, Ω_1^* and Ω_2^* lie on the negative real semiaxis. On increasing the modulus of the external perturbation frequency, the roots move towards each other. When $|W| > |W|_1$ where $|W|_1$ denotes some critical value, the roots move away from the real axis and are distributed symmetrically about it on both sides. This is caused by the oscillatory character of the function $\mathrm{Ai}'(\Omega)$ at $\Omega = |\Omega| e^{i\pi}$. Denoting by a_j^* the values of Ω at which $\mathrm{Ai}'(\Omega)$ reaches its consecutive maxima (Ω is real and varies from zero to $-\infty$), then the consecutive critical values $|W|_j$ can be found from the formula

$$|W|_j = a_j^* \left[\frac{\mathrm{Ai}'(a_j^*)}{I_1(a_j^*) - I_0}\right]^{1/2}$$

From (2.1) we find that when $|\Omega| \gg 1$, $\pi/3 < \arg W \leqslant \pi/2$, then the dispersion equation has the following sequence of zeros:

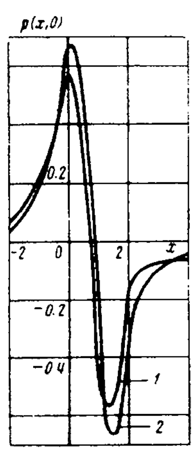

Fig.4

$$\Omega_n^* \approx -\left[\frac{3}{2}\pi\left(n + \frac{1}{4}\right)\right]^{2/3} + (-1)^n \frac{|W|^2}{[3/2\,\pi(n + 1/4)]^{1/3}} \exp(2i \arg W) \qquad (2.2)$$

distributed either along the negative real semiaxis (if $\arg W > \pi/2$ /4,5/), or alternately on each side of it. Moreover, the further the root from the coordinates origin, the nearer it approaches to the semiaxis $\arg \Omega = \pi$. When $\arg W = \pi/3$, the domain of admissible values of Ω contains only those terms of the sequence Ω_n^* which satisfy the condition $\mathrm{Im}\, \Omega > 0$. In the plane $k = k_1 + ik_2$ the roots have the corresponding sequence k_n^* distributed in the second quarter, and such that

$$|k_n^*| = O\left(\frac{1}{n}\right), \quad \lim_{n \to \infty} \arg k_n^* = -2\pi + \frac{3}{2} \arg W \qquad (2.3)$$
$$(\pi/3 \leqslant \arg W \leqslant \pi/2)$$

In the region Ω not containing the negative real semiaxis the function $P(k)$ can be written, for $|\Omega| \gg 1$, as follows:

$$P(k) = ikf(k) \frac{\Omega^3}{\Omega^3 - W^3} = ikf(k) \frac{W}{W + k^2} \qquad (2.4)$$

If $0 \leqslant \arg W < \pi/3$, then the expression (2.4) holds over the whole domain of admissible values of $\Omega\ (|\Omega| \gg 1)$.

Let now $|W| \gg 1$. In this case the dispersion equation has two solutions in the region $|\arg\Omega| < \pi$ which appear, in the plane k, in the upper and lower half-plane respectively

$$k_{1l}^{*} \approx |W|^{1/2} \exp\left(-\frac{3\pi i}{2} + \frac{i}{2}\arg W\right) \tag{2.5}$$

$$k_l^{**} \approx |W|^{1/2} \exp\left(-\frac{\pi i}{2} + \frac{i}{2}\arg W\right),$$

$$(|W| \gg 1, 0 \leqslant \arg W \leqslant \pi/2)$$

3. Structure of the solution of the problem in the physical plane.

The unknown function of pressure can be obtained in the physical plane using the formula

$$p\ (t, x) = \mathrm{Re}\ [P\ (x) \exp\ (Wt)] \tag{3.1}$$

where $P(x)$ is given by (1.5). Knowing the form of the function $f(k)$, we use the Jordan Lemma, the theorem of residues and the properties of the dispersion relation to obtain

$$P(x) = \begin{cases} -B(W, k^{**}) \exp(ik^{**}x), & x < 0 \\ \sum\limits_n B(W, k_n{}^{*}) \exp(ik_n{}^{*}x) - I_- + I_+, & x > 0 \end{cases} \tag{3.2}$$

$$B(W, k) = -kf(k)\frac{d\mathrm{Ai}(\Omega)}{d\Omega}\left\{\frac{d}{dk}\left[\frac{d\mathrm{Ai}(\Omega)}{d\Omega} + (ik)^{4/3}(I_0 - I_1(\Omega))\right]\right\}^{-1}$$

where I_+ and I_- are integrals along the right and left edge of the cut respectively. If, in the case of harmonic oscillations of the vibrator ($\arg W = \pi/2$) an infinite train of pressure perturbations always propagates downstream, then in the case of the oscillations with increasing amplitudes, and for the specified ratio of the frequencies, namely for $\omega_2/\omega_1 < \sqrt{3}$, the number of waves moving downstream becomes finite.

Figure 4, curves 1 and 2, depict the form of the function $p(0, x)$ for $|W| = 1, \arg W = \pi/2, \arg W = \pi/6$, respectively ($b = 1, a = 2, h = 2$). Reduction in the value of $\arg W$ is accompanied by a slight increase in the amplitude of the function $P(x)$, and its more rapid decay as $x \to +\infty$. This agrees well with the asymptotic estimate obtained using the method given in /2/.

$$P\ (x) = O\ (x^{-1} \exp\ [-2^{2/3}3^{-1/2}\ |W|^{3/4}x^{1/2} \cos\ (^3/_4 \arg W)]), \qquad (x \to +\infty)$$

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